APPENDIX
Multivariate OI correlation functions

The optimal interpolation procedure, adapted from (Bretherton et al. 1976), has the goal to map pressure P and velocity (u, v) from limited observations with noise, and constrained, in this study, to satisfy geostrophic balance. The best linear combination of input measurements, ϕ (which in our case is multivariate, P, u, and v on measurement sites), is sought to estimate the physical variable, θ (which may also be either P, u, or v, on *any* location within or near the measurement array), taking into account statistical cross-covariance properties between the variables. This can be expressed as:

$$\hat{\theta} = \sum_{s=1}^{N} b_s \phi_s = b^T \phi,$$

where the weights, b, at N input sites (labeled $s = 1, \cdots, N$) are selected to minimize the expected error,

$$\mathcal{E} = E\{(\hat{\theta} - \theta)^2\}.$$

θ and ϕ are assumed to be random variables with zero mean. $E\{\}$ denotes the expectation value operator. The result, which may be termed the Gauss-Markov theorem, is

$$\hat{\theta} = CA^{-1}\phi,$$

where $b = CA^{-1}$, and $A = E\{\phi\phi^T\}$ is the covariance matrix between inputs, with elements $A_{sr} = E\{\phi_s\phi_r\}$, and $C = E\{\theta\phi^T\}$ is the cross-covariance row-vector between the output variable and each of the inputs, with elements $C_r = E\{\theta\phi_r\}$. The error field is

$$\mathcal{E} = E\{\theta^2\} - CA^{-1}C^T.$$

The weights b and the errors $\mathcal{E}$ are determined entirely from the statistics, *i.e.*, based on the covariances as a function of only the measurement locations.

An important property of the above linear estimator is that the optimal estimate of a linear operator $\mathcal{L}$ on θ may be found by applying the operator on the the optimal estimate of θ, *i.e.*

$$\hat{\theta}_{\mathcal{L}} = \mathcal{L}_{\mathbf{x}}[\hat{\theta}(\mathbf{x})].$$

It follows that $\hat{\theta}_{\mathcal{L}}$ can be found from the *same* input data by applying the linear operator to the covariance vector, (*i.e.,* simply by redefining the covariance function C), as follows[1]:

$$\hat{\theta}_{\mathcal{L}} = \mathcal{L}_{\mathbf{x}}[\hat{\theta}(\mathbf{x})] = \mathcal{L}_{\mathbf{x}}[C(\mathbf{x}, \mathbf{y})\ A^{-1}(\mathbf{y_1}, \mathbf{y_2})\ \phi(\mathbf{y})] = C_{\mathcal{L}} A^{-1}\phi.$$

Hence the new cross-covariance function to estimate $\hat{\theta}_{\mathcal{L}}$ is

$$C_{\mathcal{L}} \equiv E\{\mathcal{L}_{\mathbf{x}}[\theta(\mathbf{x})]\ \phi^T(\mathbf{y})\} = \mathcal{L}_{\mathbf{x}}[C(\mathbf{x}, \mathbf{y})]. \tag{1}$$

[1]Here the different position variable names inside the parentheses are used to distinguish the position variables associated with different physical variables. For example, $\phi(\mathbf{y_2})$ indicates that the matrix element ϕ_s depends on a position (site) variable $\mathbf{y_2}$; $A(\mathbf{y_1}, \mathbf{y_2})$ indicates that the matrix element $A_{rs} = E\{\phi_r(\mathbf{y_1})\phi_s(\mathbf{y_2})\}$ depends on two position variables $\mathbf{y_1}, \mathbf{y_2}$ associated with ϕ_r and ϕ_s respectively. The subscript of the operator is also used to indicate its dependent variable.

As an example of using this extension, the geostrophic constraint may be imposed *via* the covariance functions. If we let θ be the velocity component u or v, and ϕ be pressure P, the mapping is from pressure to velocity (designated "p2u"). The linear operators for the above formulae are defined from $u = -(1/f\rho)\partial_y\phi = \mathcal{L}^u\phi$, or $v = (1/f\rho)\partial_x\phi = \mathcal{L}^v\phi$. So the proper shorthand expressions for mapping 'p2u' are

$$\hat{u} = (\mathcal{L}^u C)A^{-1}\phi$$
$$\hat{v} = (\mathcal{L}^v C)A^{-1}\phi.$$

Note that these estimated $\hat{u}$ and $\hat{v}$ *exactly* satisfy the geostrophic constraint regardless of the noise in the input data ϕ on which they depend, since the algorithm is equivalent to first estimating a streamfunction and then applying $\mathcal{L}^u$ and $\mathcal{L}^v$ on it.

A related fact is that the covariance function between outputs, $\mathcal{L}_{\mathbf{y_1}}[\eta(\mathbf{y_1})]$ and $\mathcal{L}_{\mathbf{y_2}}[\xi(\mathbf{y_2})]$, of two linear operators[2] can be found by applying the two operators to the covariance function between two input variables, $\eta(\mathbf{y_1})$ and $\xi(\mathbf{y_2})$. This can be expressed as

$$E\{\eta_{\mathcal{L}_{\mathbf{y_1}}}\xi_{\mathcal{L}_{\mathbf{y_2}}}\} = E\{\mathcal{L}_{\mathbf{y_1}}\eta(\mathbf{y_1})\mathcal{L}_{\mathbf{y_2}}\xi(\mathbf{y_2})\} = \mathcal{L}_{\mathbf{y_1}}\mathcal{L}_{\mathbf{y_2}}E\{\eta(\mathbf{y_1})\xi(\mathbf{y_2})\}. \tag{2}$$

Note Eqn (1) is a special case of the above relation where $\mathcal{L}_{\mathbf{y_2}}$ is an identity operator.

For the present study, a Gaussian covariance function was fitted to the observed pressure data

$$E\{P(x_1, y_1)\ P(x_2, y_2)\} = \sigma_P^2 \exp(-r^2/r_0^2),$$

where σ_P^2 is the pressure variance, $r = \{(x_1 - x_2)^2 + (y_1 - y_2)^2\}^{\frac{1}{2}}$, and r_0 is a chosen correlation length. For convenience, we define a streamfunction $\psi = \frac{P}{f\rho}$. The autocovariance function of ψ is then

$$E\{\psi(x_1, y_1)\ \psi(x_2, y_2)\} = \sigma_\psi^2 \exp(-r^2/r_0^2). \tag{3}$$

Table 1 lists all the relevant covariance functions that are determined using Eqns (2) and (3). As an example, $E\{u(x_1, y_1)v(x_2, y_2)\}$ in matrix A, needed for mapping from velocity to pressure (designated "u2p") can be found by

$$\begin{aligned} E\{u(x_1, y_1)\ v(x_2, y_2)\} &= E\{-\frac{\partial\psi(x_1, y_1)}{\partial y_1}\ \frac{\partial\psi(x_2, y_2)}{\partial x_2}\} \\ &= -\frac{\partial}{\partial y_1}\ \frac{\partial}{\partial x_2}\ E\{\psi(x_1, y_1)\ \psi(x_2, y_2)\} \\ &= 4(x_2 - x_1)(y_2 - y_1)r_0^{-4}\sigma_\psi^2 \exp(-r^2/r_0^2). \end{aligned}$$

Note that the constant σ_ψ^2 appears in the numerator and denominator of the product CA^{-1}, thus canceling out. Hence it is conventional to work with normalized covariance functions, treating $\sigma_\psi^2 = 1$.

In the last paragraph as well as in Table 1, we did not take into account the noise in the measurements. We now assume that the noise at a measurement site is neither correlated with measured signals nor with the noise at other measurement sites. Thus, the noise contribution only appears on

[2]The notation $\mathcal{L}_{\mathbf{y_1}}$ and $\mathcal{L}_{\mathbf{y_1}}$ is meant to imply two different operators operating on different variables. It is usually clear from the context whether the two operators are the same or different, so this notation avoids introducing more complicated symbols to indicate a different linear operator.

Table 1: Covariance functions of streamfunction and current. α_x and α_y are defined respectively by $x_2 - x_1 = r\cos\alpha_x$ and $y_2 - y_1 = r\cos\alpha_y$.

	$\psi(x_2, y_2)$	$u(x_2, y_2)$	$v(x_2, y_2)$
$\psi(x_1, y_1)$	$F(r) = \sigma_\psi^2 \exp\left(-(\frac{r}{r_0})^2\right)$	$r\cos\alpha_y \cdot 2r_0^{-2}F(r)$	$-r\cos\alpha_x \cdot 2r_0^{-2}F(r)$
$u(x_1, y_1)$	$-r\cos\alpha_y \cdot 2r_0^{-2}F(r)$	$(1 - 2(\frac{r}{r_0})^2\cos^2\alpha_y) \cdot 2r_0^{-2}F(r)$	$2(\frac{r}{r_0})^2\cos\alpha_x\cos\alpha_y \cdot 2r_0^{-2}F(r)$
$v(x_1, y_1)$	$r\cos\alpha_x \cdot 2r_0^{-2}F(r)$	$2(\frac{r}{r_0})^2\cos\alpha_x\cos\alpha_y \cdot 2r_0^{-2}F(r)$	$(1 - 2(\frac{r}{r_0})^2\cos^2\alpha_x) \cdot 2r_0^{-2}F(r)$

the diagonal entries of the matrix A. Specifically, if A_0 is the matrix whose elements are derived from the entries in Table 1, then the covariance matrix between inputs is

$$A = A_0 + \mathcal{E}_0,$$

where $\mathcal{E}_0$ is a diagonal matrix whose s^{th} diagonal element is the noise variance in the s^{th} input variable. Note the matrix C will not be affected by the noise because of the assumption that the noise in the input is not correlated with the output signal.

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