3.6 Wave-action equation

As in the case of deep water, it can be shown that the *wave action density* is always conserved as waves propagate as long as there are no source/sink of wave energy. The wave action density is defined as

$$N(k_x, k_y) = \frac{1}{\omega} G(k_x, k_y). \tag{47}$$

With forcing terms, the *wave action density* equation can be expressed as

$$\frac{\partial N}{\partial t} + \frac{\partial N}{\partial x_i} \cdot \frac{\partial \sigma}{\partial k_i} - \frac{\partial N}{\partial k_i} \cdot \frac{\partial \sigma}{\partial x_i} = F_\beta + F_d + F_n \tag{48}$$

where k_i, $(i = 1, 2)$ is the wavenumber vector. The apparent frequency σ is defined as

$$\sigma = \omega + k_i U_i \tag{49}$$

where U_i, $(i = 1, 2)$ is the surface current vector.

The source term F_β can be described as

$$F_\beta = \beta_p N \tag{50}$$

when the wave amplitude is small.

For deep water gravity waves the wave dissipation is dominated by wave-breaking. For finite depth water the wave breaking dissipation is modified, and the bottom dissipation is added.

The nonlinear interaction term can be written, in general, as

$$\begin{aligned} F_n &= \int\int\int Q[N(\mathbf{k}), N(\mathbf{k_1}), N(\mathbf{k_2}), N(\mathbf{k_3})] \\ &\times \delta(\omega + \omega_1 + \omega_2 + \omega_3)\delta(\mathbf{k} + \mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3}) d\mathbf{k_1} d\mathbf{k_2} d\mathbf{k_3} \end{aligned} \tag{51}$$

for the 4 wave interaction. In addition, the 3 wave interaction is added for very shallow water conditions.

3.7 Finite depth effect on wave action conservation

3.7.1 Conservation of wave action of a single wave train

When a wave train with an wave amplitude $|A|$ propagates in a region with a slowly varying water depth and there are no external forcing (wind input, dissipation) or currents, the conservation of the *wave action* can be expressed as

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x_i} \cdot (C_{g_i} N) = 0 \tag{52}$$

where the wave action is defined as

$$N = E/\omega \tag{53}$$

and $E = \rho g |A|^2/2$ is the energy of the wave train. The wavenumber and the frequency are constrained such that

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0, \tag{54}$$

and the two components of the wavenumber are related by

$$\frac{\partial k_x}{\partial y} = \frac{\partial k_y}{\partial x}. \tag{55}$$

For example, if the wave field is independent of time, from (54)

$$\frac{\partial \omega}{\partial x_i} = 0, \tag{56}$$

i.e., the frequency is constant. Therefore,

$$\omega^2 = gk \tanh kh = constant. \tag{57}$$

This, together with (55), determines how the wavenumber k_i varies as the water depth varies in x_i.

If, the water depth and the wave field vary in x only, i.e., are independent of t and y, and the wave train propagates in x, the wavenumber $k = k_x$ increases as h decreases, i.e., the waves become shorter. From (52) $C_g N$ is also conserved, i.e., independent of x. Since

$$E = \frac{1}{2}\rho g |A|^2 = \omega N = \frac{\omega}{C_g}(C_g N) \tag{58}$$

and ω is constant, the wave energy is inversely proportional to C_g. If the group velocity is normalized by the deep water value $g/2\omega$,

$$\frac{C_g}{C_g|_{z=\infty}} = \frac{2\omega}{g} C_g = \tanh kh \left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{59}$$

i.e., the normalized group velocity varies with the normalized depth kh. As kh decreases, the group velocity first increases by 20% (around $kh \sim 1.2$) and then decreases rapidly. Therefore, the wave amplitude first decreases by 10% and then increases rapidly.

3.7.2 Conservation of wave action density

When a wave field with the wave spectrum $G(k_i, x_i, t)$ propagates in a region with a slowly varying water depth $h(x_i)$ and there are no external forcing (wind input, dissipation, nonlinear wave interactions) or currents, the conservation of the *wave action density* can be expressed as

$$\frac{\partial N}{\partial t} + \frac{\partial N}{\partial x_i} \cdot \frac{\partial \omega}{\partial k_i} - \frac{\partial N}{\partial k_i} \cdot \frac{\partial \omega}{\partial x_i} = 0. \tag{60}$$

The term $-\frac{\partial \omega}{\partial x_i}$ indicates how the wavenumber k_i varies as a particular wave packet propagates. Since the dispersion relation is

$$\omega^2 = gk \tanh kh, \tag{61}$$

we may differentiate this by x_i, yielding

$$2\omega \frac{\partial \omega}{\partial x_i} = \frac{\partial}{\partial h}(gk \tanh kh) \frac{\partial h}{\partial x_i} = \frac{gk^2}{\cosh^2 kh} \frac{\partial h}{\partial x_i}. \tag{62}$$

Therefore, when h decreases, the sign of $-\frac{\partial \omega}{\partial x_i}$ is positive, as expected.

3.8 Interaction between waves and currents with a finite depth

3.8.1 Conservation of wave action of a single wave train

When a wave train with an wave amplitude $|A|$ propagates in a slowly varying current $U(x_i, t)$ and there are no external forcing (wind input, dissipation), the conservation of the *wave action* can be expressed as

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x_i} \cdot [(C_{g_i} + U_i)N] = 0 \tag{63}$$

where the wave action is defined as

$$N = E/\omega \tag{64}$$

and $E = \rho g|A|^2/2$ is the energy of the wave train. The wavenumber and the frequency are constrained such that

$$\frac{\partial k_i}{\partial t} + \frac{\partial \sigma}{\partial x_i} = 0 \tag{65}$$

with the apparent frequency

$$\sigma = \omega + k_i U_i. \tag{66}$$

For example, if the wave field and the current are independent of time, from (65)

$$\frac{\partial \sigma}{\partial x_i} = \frac{\partial}{\partial x_i}(\omega + k_i U_i) = \frac{\partial k_i}{\partial x_i}\frac{\partial \omega}{\partial k_i} + \frac{\partial h}{\partial x_i}\frac{\partial \omega}{\partial h} + \frac{\partial}{\partial x_i}(k_i U_i) = 0, \tag{67}$$

i.e., the apparent frequency is constant. This determines how the wavenumber varies as the surface current and the water depth h vary in x_i.

3.8.2 Conservation of wave action density

When a wave field with the wave spectrum $G(k_i, x_i, t)$ propagates in a slowly varying current $U(x_i, t)$ and there are no external forcing (wind input, dissipation, nonlinear wave interactions), the conservation of the *wave action density* can be expressed as

$$\frac{\partial N}{\partial t} + \frac{\partial N}{\partial x_i} \cdot \frac{\partial \sigma}{\partial k_i} - \frac{\partial N}{\partial k_i} \cdot \frac{\partial \sigma}{\partial x_i} = 0 \tag{68}$$

where

$$N = \frac{G}{\omega}. \tag{69}$$

The apparent frequency σ is defined as

$$\sigma = \omega + k_i U_i. \tag{70}$$

Here,

$$\frac{\partial \sigma}{\partial k_i} = \frac{\partial \omega}{\partial k_i} + U_i = C_{g_i} + U_i \tag{71}$$

is the (apparent) speed of the wave packet propagation and is the sum of the group velocity and the current. The term

$$-\frac{\partial \sigma}{\partial x_i} = -\frac{\partial \omega}{\partial h}\frac{\partial h}{\partial x_i} - k_i \frac{\partial U_i}{\partial x_i} \tag{72}$$

indicates how the wavenumber k_i varies as a particular wave packet propagates.

When the forcing terms are included, the action density equation becomes

$$\frac{\partial N}{\partial t} + \frac{\partial N}{\partial x_i} \cdot \frac{\partial \sigma}{\partial k_i} - \frac{\partial N}{\partial k_i} \cdot \frac{\partial \sigma}{\partial x_i} = F_\beta + F_n + F_d \tag{73}$$

(1) If the terms on the left hand side are relatively small, the three terms on the right must balance one another, i.e.,

$$0 = F_\beta + F_n + F_d \tag{74}$$

Therefore, the effect of the variable current/depth is negligible.

(2) If the terms on the left hand side are relatively large, the three terms on the right are negligible, i.e., the effect of the variable current/depth is the strongest.

(3) If the terms on the left hand side are comparable to the terms on the right, the current/depth effect is present but is weakened (relaxation).